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Technical note 11-7

The energy approach of fracture of beams by joints loaded perpendicular to the grain

Derivation of the basic equations

Introduction

In the CIB-W18 publication: CIB-W18/33-7-7, the full derivation of the $\sqrt{n/n_c}$ -factor of the Compliance method was not given but kept for discussion with the interested. Also the derivation of the general compliance equation of joints in the middle of the beam, including the influence of the change of the normal force at a crack increment, was not given but only was send afterwards to the interested people.
Both derivations are given here now as supplement on the CIB-W18/33-7-7 publication and it is important to know that the theory explains precisely all known data at that time as for instance of the investigations of the Karlsruhe University.

Energy approach

When a cantilever beam is cracking by a constant load V, giving a deflection increase of δ at V, then the applied energy to the beam is V·δ. The energy balance equation then is:

$$V\delta = V\delta/2 + E_c \quad (1)$$

where $V\delta/2$ is the increase of the elastic energy and E_c the energy of crack extension. From eq.(1) follows that:

$$E_c = V\delta/2 \quad (2)$$

Thus the energy of crack extension is equal to the increase of elastic energy.
Eq.(2) also can be written with de incremental deflection δ= du:

$E_c = V^2 d(u/V)/2 = G_f bhd(\beta)$ or:

$$V = \sqrt{\frac{2G_f bh}{\partial(u/V)/\partial\beta}} \quad (3)$$

In this equation is G_f the fracture energy per unit crack surface, (called energy release rate) and is the crack surface increase "bhd(β)" where "b" is the width of the beam and "h" the height and the crack length $l = \beta h$ is related to "h".
When the load on the cantilever beam, mentioned above, is prevented to move, the energy balance, eq.(1) becomes:

$0 = E_e + E_c$, or: $E_c = -E_e = -V\delta / 2$ (4)

for the same crack length and now the energy of crack extension is equal to the decrease of elastic energy in the beam.

When the joint at load V becomes determining and just flows at δ_1 when splitting of the beam occurs, then eq.(1) becomes:

$$V\delta = (V\delta_1)/2 + V(\delta - \delta_1) + E_s \quad (5)$$

where again $V\delta/2$ is the increase of the elastic energy and $V(\delta - \delta_1)$ the plastic energy of the joint. From eq.(5) follows that:

$$E_c = V\delta_1 / 2 \quad (6)$$

the same as eq.(2), despite of the plastic deformation.

For connections, plastic deformation in the last case will not occur because it is coupled with crack extension. When the dowels of the joint are pressed into the wood, the crack opening increases and thus also crack extension. It can be seen in eq.(5), that when flow occurs, the total applied energy Vδ is used for the plastic deformation. This is a comparable situation as given by eq.(4), and the at the plastic flow coupled crack extension will cause a decrease of the elastic energy. Eq.(5) thus is for joints:

$$V\delta = (V\delta_1 - \delta_2)/2 + V(\delta - \delta_1) + E_s \quad (7)$$

where $V\delta_2 / 2$ is the decrease of the elastic energy by the part of crack extension due to the plastic deformation. From eq.(7) now follows:

$$E_s = V(\delta_1 + \delta_2)/2 \quad (8)$$

and eq.(3) becomes:

$$V = \sqrt{\frac{2G_f bh}{\partial((u_1 + u_2)/V)/\partial\beta}} \quad (9)$$

From eq.(6) and (8) follows that $V_c\delta_{1c} = V(\delta_1 + \delta_2)$, where $V_c\delta_{1c}$ is the part according to eq.(6) when the connection is as strong as the beam. Thus:

$$\frac{\delta_1 + \delta_2}{\delta_{1c}} = \frac{V_c}{V} = \frac{n_c V_n}{n V_n} = \frac{n_c}{n} \quad (10)$$

where V_n is the ultimate load of the dowel at flow.

Substitution of eq.(10) into eq.(9) gives:

$$V = \sqrt{\frac{2G_f bh}{\partial(u_{1c}/V)/\partial\beta} \cdot \frac{n}{n_c}} \quad (11)$$

what is equal to $\sqrt{n/n_c}$ times the strength according to eq.(3) for $u = u_{1c}$, thus $\sqrt{n/n_c}$ times the splitting strength of the beam.

According to eq.(7), the theoretical lower bound of V according to eq.(11) occurs at $\delta_1 = \delta_2$, Thus when n/n_c > 1/2. In paper CIB-W18/33-7-7 a value of 0.5 to 0.4 is mentioned according to the data giving for small amount of dowels:

$$V = \sqrt{\frac{2G_f bh}{\partial(u_{1c}/V)/\partial\beta}} \cdot \sqrt{0,45} = \sqrt{\frac{2G_f bh}{\partial(u_{1c}/V)/\partial\beta}} \cdot 0.67 \qquad (12)$$

because below this boundary eq.(7) applies with $\delta_1 = \delta_2$, giving:

$$E_s = V\delta_1 \qquad (13)$$

as condition or crack extension coupled to plastic flow of the connection. In this case ,V increases enormously by the possible hardening perpendicular to the grain by the spreading effect of embedding strength, what is discussed in the mentioned paper. Thus splitting occurs when $V\delta_1$ reaches the critical value of E_s, thus at a value of $\sqrt{GG_f}$ of 0,67·18 = 12 $N/mm^{1.5}$ according to eq.(12) as also follows from the measurements. On this value the proposal for the Eurocode is based. .

Derivation of the compliance equation of u_{1c}/ V

Because the measurements did not show a difference in strength and fracture energy of joints at the end or in the middle of the beam, the influence of the normal force increment was neglected at crack extension in the mentioned paper CIB-W18/33-7-7. For end joints, the split off part is unloaded and there is no normal force and the situation and fracture equations are the same as for the notched beams discussed in the downloadable publication on this site: A New and Consistent Theory of Fracture Mechanics of Wood in chapter 4. For joints in the middle of the beam, splitting goes in the direction of lower moments and is stable until the totally splitting of the beam. Equilibrium with this state of splitting over the whole length of the beam was regarded to be critical end state in the analysis of paper CIB-W18/33-7-7. The theory predicted a hardening behaviour until only shearing work remains, explaining the same strength for end- and middle joints. It always is possible to omit negligible terms to obtain simple and safe Code rules. However the proof that this neglect is right is given in the following derivation of the exact equation.

The analysis can be based on the compliance difference of the cracked and un-cracked state. When half a beam is regarded, as given in fig. 1, that is loaded by a constant V and starts cracking, the deflection at V increases with δ (see fig. 2) and work is done by the force V of 2ΔW = V·δ, what is twice the increase of the strain energy (ΔW = V·δ/2) of the beam and therefore ΔW is used to increase the strain energy and the other equal amount of ΔW is used as fracture energy. Because δ is the difference of the cracked and "un-cracked" state, only the deformation of the cracked part βh minus the deformation of that same part βh in the un-cracked state, need to be calculated, because the deformation of all other parts of the beam by load V are the same in cracked and un-cracked state.

The deflection δ can be calculated from elementary beam theory of elasticity. The stress distribution then can be seen to be the first term of a polynomial row

expansion of the real stress distribution. Only the first term of the row contributes to the deflection. The other terms, representing the possible difference from this elementary stress distribution, form an internal equilibrium system, causing no, or negligible deflection of the beam and thus also the shear distribution can be taken to be parabolic according to this elementary beam theory, as only component contributing to the deflection. It thus is not right to regard an additional deformation δ_r, as is done, due to the non-linearity and clamping effect of the cantilevers βh, formed by the crack. The St. Venant length of this clamping effect at the end of a crack length l is independent of the stress level and thus is the same as at the end of a propagated crack: l + Δl and because this propagation Δl is determined by the difference of the strain energy, the same amount of clamping effect before and after propagation are subtracted from each other so that there is no contribution of this effect to the strain energy difference of crack formation. If this effect would have an influence, there should be a difference in notched beams in the splitting force for a real notch of length βh and a vertical saw cut at a distance βh from the support, because that slit has at least twice that clamping effect.
For a connection at the middle of a beam the following applies after splitting (see fig. 1). The part above the crack (stiffness $I_2 = b(1-\alpha)^3 h^3/12$) carries a moment M_2 and normal force N and the part below the crack (stiffness $I_1 = b\alpha^3 h^3/12$) carries in the middle a moment $M_3 = V\lambda - M_1$, a normal force N and a shear force V. and at the end of the crack the moment M_1 .is negative.
The rotation φ at the end of half the crack length $\lambda = \beta h$ is for both beams 1 and 2:

$$\varphi = \frac{M_2\lambda}{EI_2} = \frac{-M_1\lambda}{EI_1} + \frac{V\lambda^2}{2EI_1}. \tag{14}$$

The normal force $N = \sigma_{c1}A_1 = \sigma_{c2}A_2$ and for compatibility:

$$\varphi = \frac{\varepsilon_2\lambda + \varepsilon_1\lambda}{h/2} = \frac{2\lambda}{Eh}\cdot\left(\frac{N}{A_2} + \frac{N}{A_1}\right) = \frac{2\lambda N}{Eh}\cdot\left(\frac{A_1 + A_2}{A_1 A_2}\right) = \frac{2\lambda N}{Eh}\cdot\left(\frac{b\alpha h + b(1-\alpha)h}{b\alpha h\cdot b(1-\alpha)h}\right) =$$

$$= \frac{2\lambda N}{Eh}\cdot\left(\frac{1}{bh\alpha(1-\alpha)}\right) = \frac{Nh/2}{3bh^3/12}\cdot\frac{\lambda/E}{\alpha(1-\alpha)} = \frac{M_n\lambda}{3EI\cdot\alpha\cdot(1-\alpha)}, \tag{15}$$

where M_n is the moment due to the normal forces: M_n = Nh/2.
From eq.(14) and eq.(15) follows that:

$$M_2 = \frac{M_n I_2}{3\cdot I\cdot\alpha\cdot(1-\alpha)} \tag{16}$$

and from eq.(1), follows that:

$$M_1 = \frac{V\lambda}{2} - M_2\cdot\frac{I_1}{I_2} = \frac{V\lambda}{2} - \frac{M_n I_1}{3\cdot I\cdot\alpha\cdot(1-\alpha)} \tag{17}$$

The applied moment at the middle of the beam is:
$VL = M_2 + Nh/2 + M_3 = M_2 + Nh/2 + V\lambda - M_1$, or:

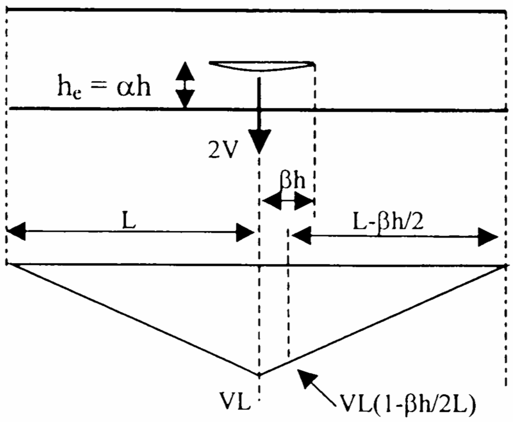

Figure 1 - Beam with a crack by the dowel force of a joint and bending moment

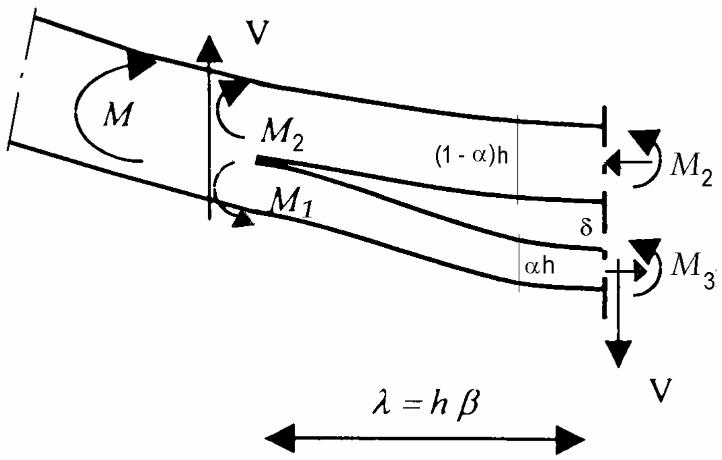

Figure 2: Statics of half the crack

$$V\cdot L = \frac{M_n I_2}{3\cdot I\cdot\alpha\cdot(1-\alpha)} + M_n + \frac{V\lambda}{2} - \frac{M_n I_1}{3\cdot I\cdot\alpha\cdot(1-\alpha)} \quad \text{or:}$$

$$V(L-\lambda/2) = M_n \frac{\alpha^3+(1-\alpha)^3+3\alpha(1-\alpha)}{3\alpha(1-\alpha)} = \frac{M_n}{3\alpha(1-\alpha)}, \text{ or:}$$

$$M_n = 3\alpha(1-\alpha)\cdot V(L-\lambda/2) \qquad (18)$$

From eq.(16) and (18) follows:

$$M_2 = V(L-\lambda/2)\cdot(1-\alpha)^3 \qquad (19)$$

and from eq.(17) and (18):

$$M_1 = V\lambda/2 - V(L-\lambda/2)\cdot\alpha^3 = -\alpha^3 VL + (\alpha^3+1)\cdot V\lambda/2 \qquad (20)$$

and $M_3 = V\lambda - M_1 = V\alpha^3 L + V\lambda(1-\alpha^3)/2$ (21)

For the change of the compliance the deflection increase at V should be known or the relative deflection of V at the support with respect to the middle of the beam.

The deflection with respect to the support upper beam 2 of length λ with constant moment M_2 is, using eq.(19) is:

$$u_2 = \phi(L-\lambda/2) = \frac{M_2\lambda}{EI_2}(L-\lambda/2) = \frac{V(L-\lambda/2)^2\cdot(1-\alpha)^3\cdot\lambda}{EI(1-\alpha)^3} = \frac{V\lambda(L-\lambda/2)^2}{EI} = u_{un}$$

what is equal to the deflection of the un-cracked beam $= u_{un}$.

For beam 1: $u_1 = \frac{M_3\lambda^2}{2EI_1} - \frac{V\lambda^2}{2EI_1}\cdot\frac{2}{3}\lambda = \frac{-V\lambda^3}{12EI\alpha^3} + \frac{V(L-\lambda/2)\lambda^2}{2EI} = -\delta + u_{un}$ (22)

what is equal to the deflection of the un-cracked beam u_{un} minus the crack opening δ. The increase of the deflection due to splitting is:

$$u_{1c} = u_1 - u_{un} = \frac{V\beta^3}{Eb\alpha^3} \quad (23)$$

With the shear deformation increase this is in total:

$$u_{1ct} = \frac{1.2}{G}\left(\frac{\beta h}{b\alpha h} - \frac{\beta h}{bh}\right) + \frac{V\beta^3}{Eb\alpha^3} \quad (24)$$

The condition of unstable equilibrium at a crack length βh is according to eq.(3):

$$V = \sqrt{\frac{2G_f bh}{\partial(u/V)/\partial\beta}} \quad (25)$$

with: $\frac{\partial(u/V)}{\partial\beta} = \frac{1.2}{bG}\left(\frac{1}{\alpha} - 1\right) + \frac{3\beta^2}{Eb\alpha^3}$ (26)

according to eq.(24). Thus eq.(25) becomes:

$$V_f = b\sqrt{\frac{GG_f h\alpha^3}{0.6\alpha^2(1-\alpha) + 1.5\beta^2 G/E}} \quad (27)$$

giving for the occurring small values of β, the in 1990 derived eq.(28):

$$\frac{V_f}{b\alpha h} = \sqrt{\frac{GG_f/h}{0.6\alpha(1-\alpha)}} \quad (28)$$

This applies because the fit of eq.(27) to the data is not better than the fit of eq.(28). Because the spreading length of the dowel forces determines the initial local tensile stress beside the dowels and this spreading length is proportional to α, β is proportional to α and of the same order, showing why the term with β can be neglected in eq.(27).

The same result can be derived immediately by assuming M_3 to be in the order of M_1. Then: $M_3 = V\lambda/2$ and the crack opening δ is:

$$\delta = \frac{1}{2}\cdot\frac{V\lambda^2}{EI_1}\cdot\frac{2}{3}\cdot\lambda - \frac{1}{2}\frac{M_1\lambda^2}{EI_1} = \frac{1}{3}\cdot\frac{V\lambda^3}{EI_1} - \frac{1}{4}\cdot\frac{V\lambda^3}{EI_1} = \frac{1}{12}\cdot\frac{V\lambda^3}{EI_1} = \frac{V\beta^3}{bE\alpha^3}$$

giving directly eq.(23). This result is to be expected. When the deformation of the un-cracked state is subtracted from the cracked deformation, then the beam is straight again except part nr. 1, that deforms as if the remaining parts of M_1 and M_3 are equal This last derivation is also given in the downloadable publication of this website: “A New and Consistent Theory of Fracture Mechanics of Wood”, chapter 5,

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